Periodic Wavelets in Walsh Analysis

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Abstract. The main aim of this paper is to present a review of periodic wavelets related to the generalized Walsh functions on the p-adic Vilenkin group $G_p$. In addition, we consider several examples of wavelets in the spaces of periodic complex sequences. The case $p = 2$ corresponds to periodic wavelets associated with the classical Walsh functions.

1. Introduction

Let $\mathbb{Z}_p$ be the discrete cyclic group of order $p$, i.e., the set $\{0, 1, \ldots, p\}$ with the discrete topology and modulo $p$ addition. The \textit{p-adic Vilenkin group} $G$ is defined to be the subgroup of $\prod_{j \in \mathbb{Z}} \mathbb{Z}_p$ consisting of sequences

$$x = (x_j) = (\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots),$$

for which there exists $k = k(x) \in \mathbb{Z}$ such that $x_j = 0$ for all $j < k$. The group operation on $G$ is denoted by $\oplus$ and defined as the coordinate-wise addition modulo $p$:

$$(z_j) = (x_j) \oplus (y_j) \iff z_j = x_j + y_j \pmod{p} \quad \text{for all } j \in \mathbb{Z}.$$ 

Let us denote the inverse operation of $\oplus$ by $\ominus$ (so that $x \ominus x = \theta$, where $\theta$ is the zero sequence). One can put a topology on $G$ as the product topology inherits from $\prod_{j \in \mathbb{Z}} \mathbb{Z}_p$.

The group $G$ is a locally compact abelian group and the sets

$$U_l := \{(x_j) \in G| x_j = 0 \text{ for } j \leq l\}, \quad l \in \mathbb{Z},$$

form a complete system neighbourhoods of the zero sequence. Notice also that

$$U_{l+1} \subset U_l \quad \text{for } l \in \mathbb{Z}, \quad \bigcap U_l = \{\theta\}, \quad \bigcup U_l = G.$$
One can show that $G$ is self-dual. The duality pairing on $G$ takes $x = (x_j)$ and $\omega = (\omega_j)$ to

$$
\chi(x, \omega) = \exp \left( \frac{2\pi i}{p} \sum_{j \in \mathbb{Z}} x_j \omega_{1-j} \right).
$$

Consider $U = U_0$ as a subgroup of $G$. This subgroup, when $p = 2$, is isomorphic to the Cantor group, which is the topological Cartesian product of countably many cyclic groups of order 2 with discrete topology. It is well-known that $U$ is a perfect nowhere-dense totally disconnected metrizable space and, therefore, $U$ is homeomorphic to the Cantor ternary set (e.g., [6, Chapter 14]). There exists a Haar measure on $G$ normalized so that the measure of $U$ is 1. For simplicity, we shall denote this measure by $dx$.

As usual, the Lebesgue space $L^2(G)$ consists of all square integrable functions on $G$. For each function $f \in L^1(G) \cap L^2(G)$, its Fourier transform $\hat{f}$,

$$
\hat{f}(\omega) = \int_G f(x) \chi(x, \omega) dx, \quad \omega \in G,
$$

belongs to $L^2(G)$. The Fourier operator

$$
\mathscr{F} : L^1(G) \cap L^2(G) \to L^2(G), \quad \mathscr{F} f = \hat{f},
$$

extends uniquely to the whole space $L^2(G)$. See [22] and [33] for further details about harmonic analysis on the group $G$.

Consider the mapping $\lambda : G \to \mathbb{R}_+$ defined by

$$
\lambda(x) = \sum_{j \in \mathbb{Z}} x_j p^{-j}, \quad x = (x_j) \in G.
$$

Take in $G$ a discrete subgroup $H = \{(x_j) \in G | x_j = 0 \text{ for } j > 0 \}$. The image of the subgroup $H$ under $\lambda$ is the set of non-negative integers: $\lambda(H) = \mathbb{Z}_+$. For each $k \in \mathbb{Z}_+$, let $h_{[k]}$ denote the element of $H$ such that $\lambda(h_{[k]}) = k$ (clearly, $h_{[0]} = \theta$). The generalized Walsh functions on $G$ can be defined by

$$
w_k(x) = \chi(x, h_{[k]}), \quad x \in G, \quad k \in \mathbb{Z}_+.
$$

So, these functions are characters for $G$. Also, it is well-known that $\{w_k | k \in \mathbb{Z}_+ \}$ is an orthonormal basis for $L^2(U)$ (when $p = 2$, we have the classical Walsh system).

Using the elements of $H$ as translations, one can study wavelets in $L^2(G)$. Orthogonal wavelets and refinable functions representable as lacunary Walsh series were introduced for the first time by Lang [24] in the context of the Cantor dyadic group and, subsequently, they have been extended and studied by several authors (see, e.g., [7]-[19], [31], [32], [37], [38]). Multiresolution analysis of functions defined on the Cantor dyadic group was studied independently by Bl. Sendov ([34]-[36]). Wavelets on the $p$-adic Vilenkin group $G$ by means of an iterative method giving rise to so-called wavelet sets were derived by J.J. Benedetto...
and R.L. Benedetto [2]. At the same time, an approach developed in [2] can be applied to wavelets on the additive group of \( p \)-adic numbers (cf. [1], [23], [25], [39]).

This paper is a continuation of our review [18], where among the main subjects are the following:

- algorithms to construct orthogonal and biorthogonal wavelets associated with the Walsh polynomials;
- estimates of the smoothness of dyadic orthogonal wavelets of Daubechies type;
- an algorithm for constructing Parseval dyadic frames.

The aim of this paper is to present a review of periodic wavelets related to the generalized Walsh functions. In Section 2, by analogy with the periodic wavelets on the line \( \mathbb{R} \) (see, e.g., [4], [5], [20], [27]-[30], [40], [41]), we define periodic wavelets on \( G \) and consider the corresponding algorithms for decomposition and reconstruction. Similar results for the case \( p = 2 \) are given in the recent papers [11] and [19]. Then, in Section 3, we use the generalized Walsh functions to define wavelets in the space \( \mathbb{C}_N \) consisting of all sequences \( x = (\ldots, x(-1), x(0), x(1), x(2), \ldots) \), such that \( x(j + N) = x(j) \) for all \( j \in \mathbb{Z} \) (cf. [3], [13], [21], [29]).

2. Periodic wavelets on the \( p \)-adic Vilenkin group

To keep our notation simple, we write \( N := p^n \) and \( \varepsilon_p := \exp(2\pi i/p) \). Define an automorphism \( A \in \text{Aut} G \) by the formula \( (Ax)_j = x_{j+1} \) for all \( x = (x_j) \in G \). Then, for \( 0 \leq k \leq N - 1 \), we let \( x_{n,k} := A^{-n}h_k \) and \( U_k^{(n)} := x_{n,k} + A^{-n}(U) \). It is easily seen that the sets \( U_k^{(n)} \) are cosets of the subgroup \( A^{-n}(U) \) in the group \( U \), and that

\[
U_k^{(n)} \cap U_l^{(n)} = \emptyset \quad \text{for} \quad k \neq l, \quad \bigcup_{k=0}^{N-1} U_k^{(n)} = U.
\]

Moreover, it is clear that \( w_l(x) \) with \( 0 \leq l \leq N - 1 \) is constant on \( U_k^{(n)} \) for each \( 0 \leq k \leq N - 1 \). We shall use the notation

\[
w_{l,k}^{(n)} := w_l(x_{n,k}) \quad \text{for} \quad 0 \leq l, k \leq N - 1.
\]

Notice that

\[
w_{l,k}^{(n)} w_{k,l}^{(n)} = \varepsilon_p^{-sq} w_{pk+sNq+l}^{(n+1)}, \quad 0 \leq s, q \leq p - 1,
\]

\[
\sum_{i=0}^{N-1} w_{i,l}^{(n)} w_{i,k}^{(n)} = \sum_{j=0}^{N-1} w_{l,j}^{(n)} w_{k,j}^{(n)} = N \delta_{l,k}, \quad 0 \leq l, k \leq N - 1.
\]
A finite sum

$$D_N(x) := \sum_{j=0}^{N-1} w_j(x), \quad x \in G,$$

is called the Walsh-Dirichlet kernel of order $N$. It is well-known that

$$D_N(x) = \begin{cases} N, & x \in U^{(n)}_0, \\ 0, & x \in U \setminus U^{(n)}_0. \end{cases}$$

Let us introduce the following spaces

$$V_n := \text{span}\{1, w_1(x), \ldots, w_{N-1}(x)\},$$

$$W^{(j)}_n := \text{span}\{w_{jN}(x), w_{jN+1}(x), \ldots, w_{(j+1)N-1}(x)\},$$

where $j = 1, \ldots, p - 1$. Note that the orthogonal direct sum of $V_n, W^{(1)}_n, \ldots, W^{(p-1)}_n$ coincides with $V_{n+1}$, that is, for $W_n := W^{(1)}_n \oplus \cdots \oplus W^{(p-1)}_n$, we have $V_n \oplus W_n = V_{n+1}$. The spaces $V_n$ and $W^{(j)}_n$ will be called the approximation spaces and wavelet spaces, respectively.

We can use the discrete Vilenkin-Chrestenson transform to recover $v \in V_n$ from the values $v(x_{n,l}), 0 \leq l \leq N - 1$. Indeed, if

$$v(x) = \sum_{k=0}^{N-1} c_k w_k(x), \quad x \in U,$$

then

$$c_k = \frac{1}{N} \sum_{l=0}^{N-1} v(x_{n,l}) w^{(n)}_{l,k}, \quad 0 \leq k \leq N - 1;$$

see, e.g., [22, Section 11.2], where the corresponding fast algorithm is given.

Suppose that $a = (a_0, a_1, \ldots, a_{N-1})$, where $a_k \neq 0$, $0 \leq k \leq N - 1$. Then we set

$$\Phi^a_n(x) := \frac{1}{N} \sum_{k=0}^{N-1} a_k w_k(x), \quad \varphi_{n,k}(x) := \Phi^a_n(x \ominus x_{n,k}), \quad 0 \leq k \leq N - 1, \quad x \in G.$$

**Proposition 2.1.** Let $v \in V_n$. Assume that

$$\alpha_{n,k} = \alpha_{n,k}(v) := \sum_{l=0}^{N-1} d_l^{-1} c_l w^{(n)}_{l,k}, \quad 0 \leq k \leq N - 1,$$

where $c_l$ are defined as in (2.4). Then

$$v(x) = \sum_{k=0}^{N-1} \alpha_{n,k} \varphi_{n,k}(x).$$

**Proof.** According to (2.2), for any $v \in V_n$ we get

$$\sum_{k=0}^{N-1} w^{(n)}_{l,k} \varphi_{n,k}(x) = a_l w_l(x), \quad 0 \leq l \leq N - 1,$$
and, in view of (2.3), (2.4) and (2.5),
\[ v(x) = \sum_{l=0}^{N-1} \sum_{j=0}^{N-1} a_l^{-1} c_l w_{1,j}^l(x) = \sum_{k=0}^{N-1} \alpha_{n,k} \varphi_{n,k}(x). \]

Therefore, the expansion in (2.6) is valid for all \( v \in V_n \).

**Remark 2.1** (cf. [40, Proposition 9]). Suppose that \( \tilde{\varphi}_{n,k} \) are defined by
\[ \tilde{\varphi}_{n,0}(x) = \sum_{j=0}^{N-1} c_j w_{1,j}(x), \quad \tilde{\varphi}_{n,k}(x) = \tilde{\varphi}_{n,0}(x \ominus x_{n,k}), \quad k = 1, \ldots, N - 1. \]

Then \( \{ \tilde{\varphi}_{n,k} \}_{k=0}^{N-1} \) is a dual shift basis for \( \{ \varphi_{n,k} \}_{k=0}^{N-1} \). Indeed, using (2.3) and (2.5), for any \( v \in V_n \) we have
\[
(v, \tilde{\varphi}_{n,k}) := \int_U v(x) \overline{\tilde{\varphi}_{n,k}(x)} \, dx \\
= \int_U \left( \sum_{l=0}^{N-1} c_l w_l(x) \right) \overline{\tilde{\varphi}_{n,0}(x \ominus x_{n,k})} \, dx \\
= \int_U \left( \sum_{l=0}^{N-1} c_l w_l(x) \right) \left( \sum_{l=0}^{N-1} a_l^{-1} w_{1,k}^l w_l(x) \right) \, dx \\
= \alpha_{n,k}(v),
\]
where the last equality follows from the orthogonality of the system \( \{ w_k | k \in \mathbb{Z}_+ \} \).

Let \( b = (b_0, b_1, \ldots, b_{pN-1}) \), where \( b_k \neq 0 \) for all \( 0 \leq k \leq pN - 1 \). In particular, we can choose
\[ b_k = \begin{cases} \frac{d_{k/p}}{p} & \text{if } k \text{ is divisible by } p, \\ 1 & \text{if } k \text{ is not divisible by } p \end{cases} \]

or
\[ b_k = \begin{cases} a_k & \text{if } k \leq N - 1, \\ 1 & \text{if } 0 \leq k \leq pN - 1. \end{cases} \]

We set
\[ \varphi_{n+1,k}(x) := \Phi^b_{pN}(x \ominus x_{n+1,k}), \quad 0 \leq k \leq pN - 1, \]
where
\[ \Phi^b_{pN}(x) := \frac{1}{pN} \sum_{k=0}^{pN-1} b_k w_k(x), \quad x \in G. \]

Then we define
\[ \psi^{(j)}_{n,k}(x) := \sum_{j=0}^{p-1} g^j_{k} \varphi_{n+1,pk+j}(x), \quad 0 \leq k \leq N - 1, \quad 1 \leq j \leq p - 1. \]

Let us show that, for each \( j \), the system \( \{ \psi^{(j)}_{n,k} \}_{k=0}^{N-1} \) is a bases for the corresponding wavelet space \( W_n^{(j)} \).
Proposition 2.2. Suppose that \( w \in W^{(j)}_n \) for some \( j \in \{1, \ldots, p-1\} \). Then

\[
w(x) = \sum_{k=0}^{N-1} \beta^{(j)}_{n,k} \psi^{(j)}_{n,k}(x),
\]

(2.7)

where, with the notation as in (2.4),

\[
\beta^{(j)}_{n,k}(w) = \sum_{l=0}^{N-1} b_{jN+l} c_{jN+l} w^{(n+1)}_{jN+l,pk}, \quad 0 \leq k \leq N-1.
\]

(2.8)

**Proof.** Let \( w \in W^{(j)}_n \) where \( j \in \{1, \ldots, p-1\} \). Then, since \( W^{(j)}_n \subset V_{n+1} \), as in Proposition 2.1 we have

\[
w(x) = \sum_{l=jN}^{(j+1)N-1} c_l w_l(x)
\]

\[
= \sum_{k=0}^{pN-1} \alpha_{n+1,k}(w) \varphi_{n+1,k}(x)
\]

\[
= \sum_{s=0}^{p-1} \sum_{k=0}^{N-1} \alpha_{n+1,sp+k}(w) \varphi_{n+1,sp+k}(x),
\]

(2.9)

where

\[
\alpha_{n+1,sp+k}(w) = \sum_{l=0}^{N-1} b_{jN+l} c_{jN+l} w^{(n+1)}_{jN+l,pk},
\]

\[
c_{jN+l} = \frac{1}{pN} \sum_{l=0}^{pN-1} w(x_{n+1,l})w^{(n+1)}_{jN+l}. \]

Here, in view of (2.1), \( w^{(n+1)}_{jN+l,pk+s} = \epsilon_p^{(n+1)} w^{(n+1)}_{jN+l,pk} \), and hence

\[
\alpha_{n+1,sp+k}(w) = \epsilon_p^{(n+1)} \alpha_{n+1,sp+k}(w), \quad 0 \leq k \leq N-1, \quad 0 \leq s \leq p-1,
\]

which by (2.8) and (2.9) yields (2.7). \( \square \)

Let \( \alpha \neq 0 \). Propositions 2.1 and 2.2 for the case where

\[
a_k = \begin{cases} 
\alpha & \text{if } k = 0 \text{ or } k = N-1, \\
1 & \text{otherwise}
\end{cases}
\]

(2.10)

can be found in [15]. In this case, we set

\[
b_k = \begin{cases} 
\alpha & \text{if } k = 0 \text{ or } k = pN-1, \\
1 & \text{otherwise}
\end{cases}
\]

Note that the value \( \alpha = 1 \) corresponds to the Haar wavelets (so, we use \( \alpha \neq 1 \) in the sequel).
For each \( l \in \{0, 1, \ldots, N-1\} \) with \( p \)-ary expansion
\[
l = \sum_{j=0}^{n-1} v_j p^j, \quad v_j \in \{0, 1, \ldots, p-1\},
\]
we let \( \gamma(l) := \sum_{j=0}^{n-1} v_j \). According to [15], in the case (2.10) we have the following equalities
\[
\varphi_{n,k}(x) = \sum_{s=0}^{p-1} \varphi_{n+1,pk+s}(x) - \frac{(1-\alpha)}{N} p^{-\gamma(k)} w_{N-1}(x),
\]  
(2.11)
\[
\varphi_{n+1,pk+s}(x) = \frac{1}{p} \left( \varphi_{n,k}(x) + \frac{1-\alpha}{\alpha N} \sum_{s=0}^{N-1} e^{\gamma/(\gamma(k))} \varphi_{n,k}(x) \right) + \frac{1}{p} \sum_{j=1}^{p-1} e^{-js} \varphi_{n,k}(x),
\]  
(2.12)
where \( 1 \leq k \leq N-1, 0 \leq s \leq p-1 \). Note also, that \( w_{N-1}(x) \) can be expressed as
\[
w_{N-1}(x) = \frac{1}{N} \sum_{s=0}^{N-1} e^{\gamma/(\gamma(k))} \varphi_{n,k}(x) = \sum_{k=0}^{N-1} \sum_{s=0}^{N-1} \gamma_{n+1,pk+s} \varphi_{n+1,pk+s}(x),
\]  
(2.13)
where \( \gamma_{n+1,pk+s} := \varphi_{N-1,pk+s}^{(N+1)} \).

For any functions \( f_n \in V_n \) and \( g_n \in W_n \) we write
\[
f_n(x) = \sum_{k=0}^{N-1} C_{n,k} \varphi_{n,k}(x), \quad g_n(x) = \sum_{j=0}^{p-1} g_{n,j}(x),
\]  
(2.14)
where
\[
g_{n,j}(x) = \sum_{k=0}^{N-1} D_{n,k} \psi_{n,k}(x),
\]
and the coefficient sequences
\[
C_n = \{C_{n,k}\}, \quad D_n = \{D_{n,k}\}, \quad 1 \leq j \leq p-1,
\]  
(2.15)
uniquely determine \( f_n \) and \( g_n \), respectively. Let us describe the algorithms, in terms of the coefficient sequences (2.15), for decomposing \( f_{n+1} \in V_{n+1} \) as the orthogonal sum of \( f_n \in V_n \) and \( g_n^{(j)} \in W_n^{(j)} \), and for reconstructing \( f_{n+1} \) from \( f_n \) and \( g_n^{(j)} \).

As a consequence of (2.12) we observe that
\[
\varphi_{n+1,pk+s}(x) = \sum_{v=0}^{N-1} A_{pk+s,v}(n) \varphi_{n,v}(x) + \sum_{j=1}^{p-1} B_{pk+s,j}(n) \psi_{n,k}(x),
\]  
(2.16)
where
\[
A_{pk+s,v}(n) = \begin{cases} 1/p + (1-\alpha)/(\alpha N), & v = k, \\ e^{\gamma/(\gamma(k))} (1-\alpha)/(\alpha N), & v \neq k \end{cases} \quad \text{and} \quad B_{pk+s,j}(n) = p^{-1} e^{-js}. 
\]
Since \( f_n + g_n = f_{n+1} \), it follows from (2.14) and (2.16) that

\[
\sum_{v=0}^{N-1} C_{n,v} \varphi_{n,v}(x) + \sum_{j=1}^{p-1} \sum_{v=0}^{N-1} D_{n,v}^{(j)} \psi_{n,v}^{(j)}(x)
\]

\[
= \sum_{s=0}^{p-1} \sum_{k=0}^{N-1} C_{n+1,pk+1,v} \varphi_{n+1,pk+s}(x)
\]

\[
= \sum_{s,k} C_{n+1,pk+s} \left\{ \sum_{v=0}^{N-1} A_{pk+s,v}^{(n)} \varphi_{n,v}(x) + \sum_{j=1}^{p-1} B_{pk+s,j}^{(n)} \psi_{n,j,v}^{(n)}(x) \right\}
\]

\[
= \sum_v \left\{ \sum_{s,k} C_{n+1,pk+s} A_{pk+s,v}^{(n)} \right\} \varphi_{n,v}(x) + \sum_{j=1}^{p-1} \left\{ \sum_{s,k} C_{n+1,pk+s} B_{pk+s,j}^{(n)} \right\} \psi_{n,j,v}^{(n)}(x).
\]

This implies that

\[
C_{n,v} = \sum_{s,k} A_{pk+s,v}^{(n)} C_{n+1,pk+s}, \quad D_{n,v}^{(j)} = \sum_{s=0}^{p-1} B_{pk+s,j}^{(n)} C_{n+1,pk+s},
\]

(2.17)

Now, using (2.11) and (2.13), we obtain

\[
\varphi_{n,v}(x) = \sum_{k=0}^{N-1} \sum_{v=0}^{p-1} Q_{pk+s,v}^{(n)} \varphi_{n+1,pk+s}(x),
\]

where

\[
Q_{pk+s,v}^{(n)} = \begin{cases} 
1 - e_p^{\gamma(k)(1-\alpha)} \gamma_{n+1,pk+s}/N, & k = v, \\
-\epsilon_p^{\gamma(k)(1-\alpha)} \gamma_{n+1,pk+i}/N, & k \neq v.
\end{cases}
\]

Therefore, we have

\[
\sum_{k,s} C_{n+1,pk+s} \varphi_{n+1,pk+s}(x)
\]

\[
= \sum_{v} C_{n,v} \left\{ \sum_{k,s} Q_{pk+s,v}^{(n)} \varphi_{n+1,pk+s}(x) \right\} + \sum_{j=1}^{p-1} \sum_{k=0}^{N-1} D_{n,k}^{(j)} \left\{ \sum_{s=0}^{p-1} e_p^{ij} \varphi_{n+1,pk+s}(x) \right\}
\]

\[
= \sum_{v} \left\{ \sum_{k,s} Q_{pk+s,v}^{(n)} C_{n,v} + \sum_{j} e_p^{ij} D_{n,k}^{(j)} \right\} \varphi_{n+1,pk+s}(x)
\]

and so

\[
C_{n+1,pk+s} = \sum_{v} Q_{pk+s,v}^{(n)} C_{n,v} + \sum_{j} e_p^{ij} D_{n,k}^{(j)}.
\]

(2.18)

We remark that the decomposition and reconstruction algorithms based on formulas (2.17) and (2.18) have more simply structure than the similar algorithms constructed in [5] for the case of trigonometric wavelets.
To conclude this section, let us consider the case where \( p = 2, N = 2^n \), and

\[
b_k = \begin{cases} 
  a_k, & 0 \leq k \leq N - 1, \\
  a_{N-k}, & N \leq k \leq 2N - 1;
\end{cases}
\]  

(2.19)

with any \( a_k \neq 0 \). Then, for all \( k \in \{0, 1, \ldots, N - 1\} \),

\[
\varphi_{n,k}(x) = \varphi_{n+1,2k}(x) + \varphi_{n+1,2k+1}(x), \quad \psi_{n,k}(x) = \varphi_{n+1,2k}(x) - \varphi_{n+1,2k+1}(x),
\]

and thus

\[
\varphi_{n+1,2k}(x) = \frac{1}{2} [\varphi_{n,k}(x) + \psi_{n,k}(x)], \quad \varphi_{n+1,2k+1}(x) = \frac{1}{2} [\varphi_{n,k}(x) - \psi_{n,k}(x)].
\]

Hence, under the condition (2.19), instead of (2.17) and (2.18) we obtain the classical Haar discrete transforms.

3. Periodic discrete \( p \)-adic wavelets

Let us denote by \( \langle k \rangle_p \) the remainder from the division of the integer \( k \) by the natural number \( p \), and let \([a]\) be the integer part of a number \( a \). For any \( a \in \mathbb{R}_+ \), the digits of the \( p \)-adic expansion

\[
a = \sum_{\nu=1}^{\infty} a_{\nu-1} p^{-\nu} + \sum_{\nu=1}^{\infty} a_{\nu} p^{-\nu}
\]

(3.1)

are defined by \( a_{\nu-1} = \langle [p^{\nu-1}a] \rangle_p \), \( a_{\nu} = \langle [p^\nu a] \rangle_p \) (so, the finite representation for a \( p \)-adic rational \( a \) is taken). We can easily see that, for each \( a \in \mathbb{R}_+ \), there exists a natural number \( \mu \) such that \( a_{\nu} = 0 \) for all \( \nu > \mu \) as well as that the first sum in (3.1) is equal to \([a]\). The representation (3.1) induces the operation of addition modulo \( p \) (or \( p \)-adic addition) on \( \mathbb{R}_+ \) as follows

\[
a \oplus_p b := \sum_{\nu=1}^{\infty} (a_{\nu-v} + b_{\nu-v}) p^{-\nu-1} + \sum_{\nu=1}^{\infty} (a_{\nu} + b_{\nu}) p^{-\nu}, \quad a, b \in \mathbb{R}_+.
\]

As usual, the equality \( c = a \oplus_p b \) means that \( c \oplus_p b = a \).

For \( N = p^t \), we set \( \mathbb{Z}_N = \{0, 1, \ldots, N - 1\} \). Suppose that the space \( \mathbb{C}_N \) consists of complex sequences \( x = (x(-1), x(0), x(1), x(2), \ldots) \), such that \( x(j + N) = x(j) \) for all \( j \in \mathbb{Z} \). An arbitrary sequence \( x \) from \( \mathbb{C}_N \) is given if the values of \( x(j) \) are given for \( j \in \mathbb{Z}_N \); therefore, the element \( x \) is often identified with the vector \((x(0), x(1), \ldots, x(N-1))\). The space \( \mathbb{C}_N \) is equipped with the following natural inner product:

\[
\langle x, y \rangle := \sum_{j=0}^{N-1} x(j) y(j).
\]

For an arbitrary \( j \in \mathbb{Z}_N \), let \( j^* \) denote the nonnegative integer defined by the condition \( j \oplus_p j^* = 0 \). For \( p = 2 \), we have \( j^* = j \), and, for \( p > 2 \), the number \( j^* \) is \( p \)-adic opposite to \( j \). For each \( x \in \mathbb{C}_N \) we denote by \( \hat{x} \) the vector from \( \mathbb{C}_N \) such that
The Vilenkin-Chrestenson functions $w^{(N)}_0, w^{(N)}_1, \ldots, w^{(N)}_{N-1}$ for the space $\mathbb{C}_N$ are defined by the equalities $w^{(N)}_k(j) = e^{|k,j|}_p$ and $w^{(N)}_k(l) = w^{(N)}_k(l + N)$, where $k, j \in \mathbb{Z}_N$, $l \in \mathbb{Z}$. For $n \geq 2$ and $p = 2$, the Vilenkin-Chrestenson functions coincide with the Walsh functions and, in the case $n = 1$ and $p \geq 2$, they are exponential functions: $w^{(p)}_k(j) = e^{|k,j|}_p$, $k, j \in \{0, 1, \ldots, p - 1\}$.

The functions $w^{(N)}_0, w^{(N)}_1, \ldots, w^{(N)}_{N-1}$ constitute an orthogonal basis in $\mathbb{C}_N$ and $||w^{(N)}_k||^2 = N$ for all $k \in \mathbb{Z}_N$. To an arbitrary vector $x$ from $\mathbb{C}_N$ the Vilenkin-Chrestenson transform assigns the sequence $\hat{x}$ of the Fourier coefficients of $x$ in the system $w^{(N)}_0, w^{(N)}_1, \ldots, w^{(N)}_{N-1}$:

$$\hat{x}(k) := \frac{1}{N} \sum_{j=0}^{N-1} x(j)w^{(N)}_k(j), \quad k \in \mathbb{Z}_N.$$

For all $x, y \in \mathbb{C}_N$, we define the $p$-convolution $x * y$ by the formula

$$(x * y)(k) := \sum_{j=0}^{N-1} x(k \oplus_p j)y(j), \quad k \in \mathbb{Z}_N.$$

By a unit $N$-periodic impulse we mean the vector $\delta_N$ from $\mathbb{C}_N$ defined by the equality

$$\delta_N(j) := \begin{cases} 1, & \text{if } j \text{ is divisible by } N, \\ 0, & \text{if } j \text{ is not divisible by } N. \end{cases}$$

The system of shifts $\{\delta_N(\cdot \oplus_p k) | k \in \mathbb{Z}_N\}$ is an orthonormal basis in $\mathbb{C}_N$ and

$$x(j) = (x * \delta_N)(j) = \sum_{k=0}^{N-1} x(k)\delta_N(j \oplus_p k), \quad j \in \mathbb{Z}_N,$$

for all $x \in \mathbb{C}_N$. For each $k \in \mathbb{Z}_N$ the $p$-adic shift operator $T_k : \mathbb{C}_N \to \mathbb{C}_N$ is defined as

$$(T_k x)(j) := x(j \oplus_p k), \quad x \in \mathbb{C}_N, \quad j \in \mathbb{Z}_N.$$ 

It follows from the definitions that, for all $x, y \in \mathbb{C}_N$, the following relations hold:

$$\langle x, y \rangle = N \langle \hat{x}, \hat{y} \rangle, \quad \hat{x} \star \hat{y} = N \hat{x} \hat{y}, \quad \langle T_k x \rangle(l) = w^{(N)}_k(l)\hat{x}(l),$$

$$\langle y, T_k x \rangle = y \star \hat{x}(k), \quad \langle T_k x, T_l y \rangle = \langle x, T_{l \oplus_p k} y \rangle, \quad k, l \in \mathbb{Z}_N.$$

For $v = 0, 1, \ldots, n$, we set $N_v = N/p^v$ and $\Delta_v = p^{v-1}$. The operators $D : \mathbb{C}_N \to \mathbb{C}_{N_v}$ and $U : \mathbb{C}_{N_v} \to \mathbb{C}_N$ given by the formulas

$$(D x)(j) := x(p j), \quad j = 0, 1, \ldots, N_v - 1,$$
Suppose that $\text{(3.2)}$

$$w$$

Calculate

The collection of vectors $u$,

Choose complex numbers $\mathbb{C}^2$.

Step 1. Theorem 3.1. The following theorem characterizes all the collections of vectors generating wavelet bases of the first stage in $\mathbb{C}_N$.

Following the approach from [21, Chapter 3], we give the following definition.

**Definition 3.1.** Suppose that $u_0, u_1, \ldots, u_{p-1} \in \mathbb{C}_N$. If the system

$$B(u_0, u_1, \ldots, u_{p-1}) = \{T_{p_k}u_0\}_{k=0}^{N_{l-1}} \cup \{T_{p_k}u_1\}_{k=0}^{N_{l-1}} \cup \cdots \cup \{T_{p_k}u_{p-1}\}_{k=0}^{N_{l-1}}$$

is an orthonormal basis in $\mathbb{C}_N$, then $B(u_0, u_1, \ldots, u_{p-1})$ is called the wavelet basis of the first stage in $\mathbb{C}_N$ generated by the collection of vectors $u_0, u_1, \ldots, u_{p-1}$.

The following theorem characterizes all the collections of vectors generating wavelet bases of the first stage in $\mathbb{C}_N$.

**Theorem 3.1.** The collection of vectors $u_0, u_1, \ldots, u_{p-1}$ generates a wavelet basis of the first stage in $\mathbb{C}_N$ if and only if the matrix

$$A(l) := \frac{N}{\sqrt{p}} \begin{pmatrix} \tilde{u}_0(l) & \tilde{u}_1(l) & \cdots & \tilde{u}_{p-1}(l) \\ \tilde{u}_0(l + N_l) & \tilde{u}_1(l + N_l) & \cdots & \tilde{u}_{p-1}(l + N_l) \\ \tilde{u}_0(l + 2N_l) & \tilde{u}_1(l + 2N_l) & \cdots & \tilde{u}_{p-1}(l + 2N_l) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{u}_0(l + (p-1)N_l) & \tilde{u}_1(l + (p-1)N_l) & \cdots & \tilde{u}_{p-1}(l + (p-1)N_l) \end{pmatrix}$$

is unitary for $l = 0, 1, \ldots, N_l - 1$.

For each $1 \leq m \leq n$ we define the following procedure for the construction of a wavelet basis of the first stage in $\mathbb{C}_N$.

**Step 1.** Choose complex numbers $b_l$, $0 \leq l \leq p^m - 1$, satisfying the condition

$$\sum_{k=0}^{p^m-1} |b_{l+kp^{m-1}}|^2 = 1, \quad l = 0, 1, \ldots, p^{m-1} - 1.$$  \hfill (3.2)

**Step 2.** Calculate $a_0, \ldots, a_{p^m-1}$ by the formulas

$$a_j = p^{-m+1/2} \sum_{l=0}^{p^m-1} b_l w_l^{(p^m)}(j), \quad j = 0, 1, \ldots, p^m - 1.$$
Step 3. Define a vector \( u_0 \in \mathbb{C}_N \), for which
\[
    u_0(j) = \begin{cases} 
        a_j, & 0 \leq j \leq p^m - 1, \\
        0, & p^m \leq j \leq p^n - 1.
    \end{cases}
\]
(3.3)

Step 4. Find vectors \( u_1, \ldots, u_{p-1} \in \mathbb{C}_N \) such that, for all \( l = 0, 1, \ldots, N_1 - 1 \), the matrix \( A(l) \) is unitary.

Using Theorem 3.1, we can verify that the resulting collection of vectors \( u_0, u_1, \ldots, u_{p-1} \) generates a wavelet basis of the first stage in \( \mathbb{C}_N \). In the case \( p = 2 \), step 4 of this procedure is carried out by the formula
\[
    u_j = (-1)^j u_0(1 + j), \quad j \in \mathbb{Z}_N,
\]
(3.4)

for \( p > 2 \), algorithms for the realization of this step were given in [28, Section 2.6] (see also [14, Section 2]). One of these algorithms is based on the Haar basis and can be described by the formulas
\[
    \hat{u}_k(l) = \frac{1}{\sqrt{2}} u_0(l + kN_1), \quad k \in \mathbb{Z}.
\]
(3.5)

The following example is obtained by modifying the orthogonal wavelets constructed for the Cantor group in [24]; it corresponds to the case \( m = p = 2 \), \( b_0 = 1 \), \( b_0 = a, b_2 = 0, b_2 = b \) in the procedure described above.

Example 3.2. Suppose that \( a \) and \( b \) are complex numbers such that \( |a|^2 + |b|^2 = 1 \).
Suppose that \( p = 2 \) and \( N \geq 4 \), and the vectors \( u_0, u_1 \in \mathbb{C}_N \) are given by the equalities
\[
    u_0(0) = \frac{1 + a + b}{2\sqrt{2}}, \quad u_0(1) = \frac{1 + a - b}{2\sqrt{2}}, \quad u_0(2) = \frac{1 - a - b}{2\sqrt{2}}, \quad u_0(3) = \frac{1 - a + b}{2\sqrt{2}},
\]
\[
    u_1(0) = \frac{1 + a - b}{2\sqrt{2}}, \quad u_1(1) = \frac{1 + a + b}{2\sqrt{2}}, \quad u_1(2) = \frac{1 - a + b}{2\sqrt{2}}, \quad u_1(3) = \frac{1 - a - b}{2\sqrt{2}},
\]
under the condition that \( u_0(j) = u_1(j) = 0 \) for \( 4 \leq j \leq N - 1 \). Then the vectors \( u_0, u_1 \) generate a wavelet basis of the first stage in \( \mathbb{C}_N \). Note that, for \( a = 1, b = 0 \), the resulting wavelet basis \( B(u_0, u_1) \) coincides with the Haar wavelet basis of the first stage described in Example 3.1.
The following two examples are similar to Examples 3 and 4 in [8].

**Example 3.3.** Suppose that \( p = 2, n > 3, \) and \( m = 3. \) We set
\[
(b_0, b_1, \ldots, b_7) = \frac{1}{2} (1, a, b, c, 0, \alpha, \beta, \gamma),
\]
where \(|a|^2 + |b|^2 = |c|^2 + |eta|^2 = |\gamma|^2 = 1.\) Then, by relation (3.3), we have
\[
u_0(0) = \frac{1}{4\sqrt{2}}(1 + a + b + c + \alpha + \beta + \gamma),
\]
\[
u_0(1) = \frac{1}{4\sqrt{2}}(1 + a + b + c - \alpha - \beta - \gamma),
\]
\[
u_0(2) = \frac{1}{4\sqrt{2}}(1 + a - b - c + \alpha - \beta - \gamma),
\]
\[
u_0(3) = \frac{1}{4\sqrt{2}}(1 + a - b - c - \alpha + \beta + \gamma),
\]
\[
u_0(4) = \frac{1}{4\sqrt{2}}(1 - a + b - c - \alpha + \beta - \gamma),
\]
\[
u_0(5) = \frac{1}{4\sqrt{2}}(1 - a + b - c + \alpha - \beta + \gamma),
\]
\[
u_0(6) = \frac{1}{4\sqrt{2}}(1 - a - b + c - \alpha - \beta + \gamma),
\]
\[
u_0(7) = \frac{1}{4\sqrt{2}}(1 - a - b + c + \alpha + \beta - \gamma).
\]
Further, we set \( u_1(j) = \nu_0(j) = 0 \) for \( 8 \leq j \leq 2^n - 1, \) and we choose the other components of the vector \( u_1 \) so that relations (3.4) are valid, i.e.,
\[
u_1(0) = -\nu_0(1), \quad u_1(1) = -\nu_0(0), \quad u_1(2) = \nu_0(3), \quad u_1(3) = -\nu_0(2),
\]
\[
u_1(4) = -\nu_0(5), \quad u_1(5) = -\nu_0(4), \quad u_1(6) = \nu_0(7), \quad u_1(7) = -\nu_0(6).
\]
The resulting pair \( u_0, u_1 \) generates a wavelet basis of the first stage in \( \mathbb{C}_N.\)

**Example 3.4.** Suppose that \( p = 3, n > 2, m = 2 \) and
\[
(b_0, b_1, \ldots, b_7) = \frac{1}{\sqrt{3}} (1, a, \alpha, 0, b, \beta, 0, c, \gamma),
\]
where \(|a|^2 + |b|^2 + |c|^2 = |\alpha|^2 + |eta|^2 + |\gamma|^2 = 1.\) Then, using (3.2) and (3.3), we obtain
\[
u_0(0) = \frac{1}{3\sqrt{3}}(1 + a + b + c + \alpha + \beta + \gamma),
\]
\[
u_0(1) = \frac{1}{3\sqrt{3}}(1 + a + \alpha + (b + \beta)e_2^2 + (c + \gamma)e_3).
\]
procedure, for in this paper becomes apparent due to the fact that, according to step 1 of the
construction of orthogonal wavelets on the Cantor and Vilenkin groups (as well
methods (see, for example, Chapters 8-10 in Mallat’s book \[\text{[26]}\], extends the range of applications of the well-known adaptive signal-approximation
example due to Lang, this pair corresponds to a linearly dependent system; see also
use (3.5) to define the other components of the vectors \( u_1, u_2 \in \mathbb{C}_N \) so that the
matrix

\[
\begin{pmatrix}
\frac{\theta}{\sqrt{3}} & \bar{u}_0(l) & \bar{u}_1(l) & \bar{u}_2(l) \\
\bar{u}_0(l + 3) & \bar{u}_1(l + 3) & \bar{u}_2(l + 3) \\
\bar{u}_0(l + 6) & \bar{u}_1(l + 6) & \bar{u}_2(l + 6)
\end{pmatrix}
\]

is unitary for \( l = 0, 1, 2 \). The resulting collection of the vectors \( u_0, u_1, u_2 \) generates
a wavelet basis of the first stage in \( \mathbb{C}_N \).

The values of the parameters \( b_i \) in Examples 3.2-3.4 are universal in the sense that they occur not only in the construction of wavelet bases in \( \mathbb{C}_N \), but also in the
construction of orthogonal wavelets on the Cantor and Vilenkin groups (as well as on the half-line \( \mathbb{R}_+ \); see [8], [10]) requires some additional constraint related to the requirement that the masks have no blocking sets (so, in Example 2, the
pair \( a = 0, b = 1 \) leads to a wavelet basis in the space \( \mathbb{C}_N \), while, in the original
example due to Lang, this pair corresponds to a linearly dependent system; see also
Example 2 in [8]). The great freedom of choice of the values of the parameters in the
construction of orthogonal wavelets in the space \( \mathbb{C}_N \) by the method described
in this paper becomes apparent due to the fact that, according to step 1 of the
procedure, for \((b_0, b_1, \ldots, b_{p-1})\) we can choose any complex vector of dimension
\( p^n \) satisfying condition (3.2) (compare with the construction of discrete Daubechies
wavelets in [3] and [21]). This property is important for applications, because it extends the range of applications of the well-known adaptive signal-approximation
methods (see, for example, Chapters 8-10 in Mallat’s book [26]).
Definition 3.2. Suppose that $m \in \mathbb{N}$, $m \leq n$. By a sequence of orthogonal wavelet filters of the $m$th stage we mean a sequence of vectors

$$u_0^{(1)} , u_1^{(1)} , \ldots , u_{p}^{(1)} , \ldots , u_0^{(m)} , u_1^{(m)} , \ldots , u_{p}^{(m)} ,$$

such that $u_\mu^{(v)} \in C_{N_{v-1}}$ for $v = 1, 2, \ldots , m$, $\mu = 0, 1, \ldots , p - 1$ and the matrices

$$A^{(v)}(l) := \frac{N_1}{\sqrt{P}} \begin{pmatrix} \tilde{u}_0^{(v)}(l) & \ldots & \tilde{u}_{p}^{(v)}(l) \\ \tilde{u}_0^{(v)}(l + N_v) & \ldots & \tilde{u}_{p}^{(v)}(l + N_v) \\ \tilde{u}_0^{(v)}(l + 2N_v) & \ldots & \tilde{u}_{p}^{(v)}(l + 2N_v) \\ \vdots & \ldots & \vdots \\ \tilde{u}_0^{(v)}(l + (p - 1)N_v) & \ldots & \tilde{u}_{p}^{(v)}(l + (p - 1)N_v) \end{pmatrix}$$

are unitary for $v = 1, 2, \ldots , m$, $l = 0, 1, \ldots , N_v - 1$.

Theorem 3.2. Suppose that the collection of vectors $u_0 , u_1 , \ldots , u_{p-1}$ generates a wavelet basis of the first stage in $C_N$. For a given $m \in \mathbb{N}$, $m \leq n$, set

$$u_\mu^{(1)}(j) = u_\mu(j), \quad u_\mu^{(v)}(j) = \Delta_1 \sum_{k=0}^{\Delta - 1} u_\mu^{(1)}(j + kN_{v-1}), \quad j \in \mathbb{Z}_{N_{v-1}}, \quad (3.7)$$

where $v = 2, \ldots , m$, $\mu = 0, 1, \ldots , p - 1$. Then the vectors

$$u_0^{(1)} , u_1^{(1)} , \ldots , u_{p-1}^{(1)} , \ldots , u_0^{(m)} , u_1^{(m)} , \ldots , u_{p-1}^{(m)},$$

constitute a sequence of orthogonal wavelet filters of the $m$th stage.

Thus, from a given vector $u_0 \in C_N$, defined by (3.2) and (3.3) we can, first, find a wavelet basis of the first stage $u_0 , u_1 , \ldots , u_{p-1}$, using (3.4) or (3.5), and then, using (3.6) obtain the sequence of orthogonal wavelet filters of the $m$th stage. Denote by $\oplus$ the direct sum of the subspaces of the space $C_N$. By the theorem that follows, from any sequence of orthogonal wavelet filters of the $m$th stage we can construct an orthonormal wavelet basis in $C_N$.

Theorem 3.3. Suppose that a sequence of orthogonal wavelet filters of the $m$th stage is given in the space $C_N$:

$$u_0^{(1)} , u_1^{(1)} , \ldots , u_{p-1}^{(1)} , \ldots , u_0^{(m)} , u_1^{(m)} , \ldots , u_{p-1}^{(m)}.$$ 

Let $\varphi^{(1)} = u_0^{(1)}$, $\psi^{(1)} = u_\mu^{(1)}$, $\mu = 1, \ldots , p - 1$, and define $\varphi^{(v)}$, $\psi^{(v)}$ for $v = 2, \ldots , m$, $\mu = 1, \ldots , p - 1$ by the formulas

$$\varphi^{(v)} = \varphi^{(v-1)} \ast U^{v-1} u_0^{(v)}, \quad \psi^{(v)} = \psi^{(v-1)} \ast U^{v-1} u_\mu^{(v)}.$$ 

Further, for $v = 1, \ldots , m$, $\mu = 1, \ldots , p - 1$, we set

$$\varphi_{-v,k} = T_{p,k} \varphi^{(v)}, \quad \psi_{-v,k}^{(\mu)} = T_{p,k} \psi^{(v)}, \quad k = 0, 1, \ldots , N_v - 1,$$
and define the subspaces

\[ V_{-\nu} = \text{span}\{\varphi_{-\nu,k}\}_{k=0}^{N_{-\nu}-1}, \quad W_{\nu}^{(m)} = \text{span}\{\psi_{-\nu,k}^{(m)}\}_{k=0}^{N_{-\nu}-1}, \]

\[ W_{-\nu} = W_{-\nu}^{(1)} \oplus \cdots \oplus W_{-\nu}^{(p-1)}. \]

Then the following expansion holds:

\[ C_N = W_{-1} \oplus W_{-2} \oplus \cdots \oplus W_{-m} \oplus V_m \quad (3.8) \]

and, for each \( \nu = 1, 2, \ldots, m \) the following properties are valid:

(a) \( V_{-\nu} = V_{-\nu-1} \oplus W_{-\nu-1} \);
(b) \( \{\varphi_{-\nu,k}\}_{k=0}^{N_{-\nu}-1} \) is an orthonormal basis in \( V_{-\nu} \);
(c) \( \{\psi_{-\nu,k}^{(1)}\}_{k=0}^{N_{-\nu}-1} \cup \cdots \cup \{\psi_{-\nu,k}^{(m-1)}\}_{k=0}^{N_{-\nu}-1} \) is an orthonormal basis in \( W_{-\nu} \).

This theorem justifies the method of constructing subspaces \( V_{-1}, \ldots, V_{-n} \) in \( C_N \) with the following properties:

(i) \( V_{-\nu-1} \subset V_{-\nu} \) for all \( \nu \in \{1, 2, \ldots, n\} \);
(ii) for each \( \nu \in \{1, 2, \ldots, n\} \), there exists a vector \( \varphi^{(\nu)} \in V_{-\nu} \) such that the system \( \{T_{p^{k}}\varphi^{(\nu)}\}_{k=0}^{N_{-\nu}-1} \) is an orthonormal basis in \( V_{-\nu} \);
(iii) for each 1 \( \leq m \leq n \), relation (3.7) is valid;
(iv) for each \( \nu \in \{1, 2, \ldots, n\} \) there exist vectors \( \psi_{1}^{(\nu)}, \ldots, \psi_{p}^{(\nu)} \in W_{-\nu} \) such that the system \( \bigcup_{n=1}^{p-1} \{T_{p^{k}}\psi_{n}^{(\nu)}\}_{k=0}^{N_{-\nu}-1} \) is an orthonormal basis in \( W_{-\nu} \).

Theorems 3.1-3.3 are proved by the author in [16]. A similar construction in the space \( L^2(\mathbb{R}^d) \) is well-known and is related to the notion of multiresolution analysis. According to the terminology used in the theory of multiresolution analysis, the sequence \( \{\varphi^{(\nu)}\}_{\nu=1}^{n} \) in property (ii) it is natural to call a scaling sequence in \( C_N \).

In particular, for \( p = 2 \), \( n = 3 \), using Theorem 3.3, we obtain three orthonormal wavelet bases in \( C_0 \):

\[ \begin{align*}
\{\psi_{-1,k}^{(1)}\}_{k=0}^{1} & \cup \{\varphi_{-1,k}^{(1)}\}_{k=0}^{1} \quad (m = 1), \\
\{\psi_{-1,k}^{(2)}\}_{k=0}^{1} & \cup \{\varphi_{-2,k}^{(1)}\}_{k=0}^{1} \cup \{\varphi_{-2,k}^{(1)}\}_{k=0}^{3} \quad (m = 2), \\
\{\psi_{-1,k}^{(3)}\}_{k=0}^{1} & \cup \{\psi_{-2,k}^{(1)}\}_{k=0}^{3} \cup \{\psi_{-3,k}^{(1)}\} \cup \{\varphi_{-3,k}^{(1)}\} \quad (m = 3). 
\end{align*} \]

In the Haar case (see Example 3.1), these bases consist of the vectors

\[ \begin{align*}
\varphi_{-1,0} &= \frac{1}{\sqrt{2}}(1, 1, 0, 0, 0, 0, 0, 0), & \psi_{-1,0} &= \frac{1}{\sqrt{2}}(1, -1, 0, 0, 0, 0, 0, 0), \\
\varphi_{-1,1} &= \frac{1}{\sqrt{2}}(0, 0, 1, 1, 0, 0, 0, 0), & \psi_{-1,1} &= \frac{1}{\sqrt{2}}(0, 0, 1, -1, 0, 0, 0, 0), \\
\varphi_{-1,2} &= \frac{1}{\sqrt{2}}(0, 0, 0, 0, 1, 1, 0, 0), & \psi_{-1,2} &= \frac{1}{\sqrt{2}}(0, 0, 0, 0, 1, -1, 0, 0),
\end{align*} \]
Suppose that \( \varphi_{-1,3} = \frac{1}{\sqrt{2}} (0, 0, 0, 0, 0, 0, 1, 1) \), \( \psi_{-1,3} = \frac{1}{\sqrt{2}} (0, 0, 0, 0, 0, 1, -1, 0) \).

\[ \varphi_{-2,0} = \frac{1}{2} (1, 1, 1, 1, 0, 0, 0), \quad \psi_{-2,0} = \frac{1}{2} (1, 1, -1, 0, 0, 0, 0), \]

\[ \varphi_{-2,1} = \frac{1}{2} (0, 0, 0, 1, 1, 1, 0), \quad \psi_{-2,1} = \frac{1}{2} (0, 0, 0, 0, 1, -1, -1), \]

\[ \varphi_{-3,0} = \frac{1}{2\sqrt{2}} (1, 1, 1, 1, 1, 1, 1, 1), \quad \psi_{-3,0} = \frac{1}{2\sqrt{2}} (1, 1, 1, 1, -1, -1, -1, -1). \]

In the general case, the orthogonal projections \( P_{-v} : C_N \rightarrow V_{-v} \) and \( Q_{-v} : C_N \rightarrow W_{-v} \) act by the formulas

\[ P_{-v} x = \sum_{k=0}^{N-1} \langle x, \varphi_{-v,k} \rangle \varphi_{-v,k}, \quad Q_{-v} x = \sum_{\mu=1}^{p-1} \sum_{k=0}^{N-1} \langle x, \psi_{-v,k}^{(\mu)} \rangle \psi_{-v,k}^{(\mu)}. \] (3.9)

Suppose that \( I \) is the identity operator on \( C_N \). Setting \( P_0 = I \), \( V_0 = C_N \) and using Theorem 3.3 for any \( x \in C_N \), we obtain the equalities

\[ x = P_{-v} x + \sum_{k=1}^{v} Q_{-k} x, \quad P_{-v+1} x = P_{-v} x + Q_{-v} x, \quad v = 1, 2, \ldots, n. \]

An arbitrary vector \( x \) from \( C_N \) can be regarded as the input signal \( a_0 = x \) and, for \( v = 1, 2, \ldots, m \), we can set

\[ a_v = D(a_{v-1} \ast \tilde{u}_0^{(v)}), \quad d^{(\mu)}_{v} = D(a_{v-1} \ast \tilde{u}_\mu^{(v)}), \quad \mu = 1, \ldots, p - 1. \] (3.10)

We can easily see that the components of the vectors \( a_v \) and \( d_{v}^{(\mu)} \) are the coefficients of the expansions (3.8) for a chosen \( x \). The application of formulas (3.9) constitutes the phase of the analysis of the signal \( x \) and yields the collection of vectors

\[ d_1^{(1)}, \ldots, d_{p-1}^{(1)}, \ldots, d_1^{(m)}, \ldots, d_{p-1}^{(m)}, a_m. \] (3.11)

The inverse passage from the collection (3.10) to the original vector \( x \) constitutes the reconstruction phase and is defined by the formulas

\[ a_{v-1} = u_0^{(v)} \ast U a_v + \sum_{\mu=1}^{p-1} u_\mu^{(v)} \ast U d_\mu^{(v)}, \quad v = m, m-1, \ldots, 1. \] (3.12)

Formulas (3.9) and (3.11) specify the direct and inverse discrete wavelet transforms associated with the sequence of wavelet filters \( u_0^{(1)}, u_1^{(1)}, \ldots, u_p^{(1)}, \ldots, \), \( u_0^{(m)}, u_1^{(m)}, \ldots, u_p^{(m)}, \) and are realized by using fast algorithms (cf. [21, Section 3.2], [28, Section 4]).

**Remark 3.1.** Suppose that \( m \in \mathbb{N}, m \leq n \). For a given sequence of vectors

\[ u_0^{(1)}, \ldots, u_{p-1}^{(1)}, v_0^{(1)}, \ldots, v_{p-1}^{(1)}, \ldots, u_0^{(m)}, \ldots, u_{p-1}^{(m)}, v_0^{(m)}, \ldots, v_{p-1}^{(m)}, \] (3.13)
such that $u^{(ν)}_μ, ψ^{(ν)}_μ ∈ C_{N−1}$ for $ν = 1, 2, ..., m$, $μ = 0, 1, ..., p−1$, we introduce the matrices $A^{(ν)}(l)$ just as in Definition 3.2 and set

$$B^{(ν)}(l) := \frac{N}{\sqrt{p}} \begin{pmatrix}
\overline{v}^{(ν)}_0(l) & \cdots & \overline{v}^{(ν)}_{p−1}(l) \\
\overline{v}^{(ν)}_0(l+N_r) & \cdots & \overline{v}^{(ν)}_{p−1}(l+N_r) \\
\overline{v}^{(ν)}_0(l+2N_r) & \cdots & \overline{v}^{(ν)}_{p−1}(l+2N_r) \\
\vdots & \cdots & \vdots \\
\overline{v}^{(ν)}_0(l+(p−1)N_r) & \cdots & \overline{v}^{(ν)}_{p−1}(l+(p−1)N_r)
\end{pmatrix}^T,$$

where $T$ denotes transposition. We say that the vectors (3.12) constitute a sequence of biorthogonal wavelet filters of the $m$th stage if

$$B^{(ν)}(l)A^{(ν)}(l) = E_p, \quad ν = 1, 2, ..., m; \quad l = 0, 1, ..., N_r − 1,$$

where $E_p$ is the identity matrix of order $p$. Using this definition, we can generalize the construction given above to the biorthogonal case and, instead of Examples 3.2-3.4, obtain the discrete analogs of the corresponding examples from [12] and [14].

Remark 3.2. Suppose that $\{w_k\}_{k=0}^∞$ is the generalized Walsh system determined from the given number $p ≥ 2$ and generating an orthonormal basis in the $L^2$-space on the interval $Δ = [0, 1)$ (the case $p = 2$ corresponds to the classical Walsh system; see, for example, [1]). To each sequence $x = (x_0, x_1, ...) \in ℓ_2(Z_+)$ we assign the function $\tilde{x} := \sum_{k=0}^∞ x_k w_k$ in $L^2(Δ)$. Using this mapping instead of the Vilenkin-Chrestenson transform, we can prove analogs of Theorems 3.1-3.3 for the space $ℓ_2(Z_+)$ (compare [21, Chapter 4]) and obtain the discrete nonperiodic analogs of the wavelet bases from [8] and [14].

Further discussions and possible applications of periodic wavelets considered in this paper can be found in the works [13] and [19].

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References


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